A Continuum Mechanics for Damaged Anisotropic Solids

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This paper develops a theory of continuum damage mechanics for anisotropic solids on the basis of both the strain energy equivalence principle and the equivalent (fictitious) line crack damage modeling. The strain energy equivalence principle is used to develop the effective continuum elastic properties of a damaged solid in terms of the undamaged anisotropic elastic properties and a scalar damage variable. The equivalent line crack representation of local damage provides a means by which the effective direction of damage propagation can be identified from the local stresses and strains that are available in the course of continuum damage analysis. A scalar damage variable is defined as the effective volume fraction of a damaged zone associated with an equivalent line crack. Finally, an iterative numerical approach to continuum damage analysis is introduced.

Key Words: Continuum Damage Mechanics, Strain Energy Equivalence Principle, Equivalent Crack Damage Modeling, Damage Variable, Damage Evolution Equation

1. Introduction

A material failure process is often assumed to involve general degradation of elastic properties due to the highly localized nucleation and growth of microdefects (*i. e.*, microcracks and microvoids) and their ultimate coalescence into macrodefects. The process and result of these irreversible, energy dissipating, microstructural rearrangements are often called damage. Extensive treatments of continuum damage mechanics can be found in the books by Kachanov (1986), Lemaitre (1992), and Krajcinovic (1996).

Because of the complex nature of damage, there is no general agreement regarding the definition of damage variable(s). As Krajcinovic and Mastilovic (1995) discussed, selection of a damage variable is largely a matter of taste and convenience, and often has no obvious physical basis. Despite the non-uniqueness of damage variable definitions, there has been extensive research on continuum damage mechanics, focusing on two major subjects: the constitutive equations for damaged materials, and damage evolution laws. Current theories of continuum damage mechanics may be classified into four categories (Lee *et al.*, 1997) on the basis of: a) the type of elastic behavior of the damaged material; and b) the scalar or tensor nature of the damage variables and evolution equations. To the author's knowledge, most theories of continuum damage mechanics have been developed for initially *isotropic* solids, and the continuum damage theory by Lee et al. (1997) (hereafter referred to as, Lee's damage theory or LDT) is certainly the first that permits *anisotropic* behavior of the damaged *isotropic* material, while using a *scalar* damage variable. Since this paper is the extension of LDT to anisotropic solids, a brief review on LDT will be given in what follows.

LDT is based on both the principle of strain energy equivalence and the concept of equivalent fictitious microcrack representation of local damage. In this context, the *strain energy equivalence principle* (SEEP) means that a material volume cell (MVC) of the damaged material containing local damage and its equivalent continuum model (ECM) should contain equal strain energy when

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they are subject to identical global displacements on their boundaries. In LDT, SEEP was used to develop the effective continuum elastic properties of a damaged solid in terms of the undamaged isotropic elastic properties as well as a new definition of scalar damage variable. One should note that, since a damaged MVC contains a single microcrack, Lee's homogenization approach differs substantially from the so called self-consistent method (*e. g.,* Budiansky and O'Connell, 1976), in which a representative volume element is assumed to contain a large number of microcracks, the statistical distribution of which is known. A single scalar damage variable, D, was defined as the effective volume fraction of a damaged zone associated with an equivalent microcrack. This definition of the damage variable was consistently used to develop a consistent damage evolution equation, in conjunction with Paris's crack growth law (Paris and Erdogan, 1963) commonly used in fracture mechanics.

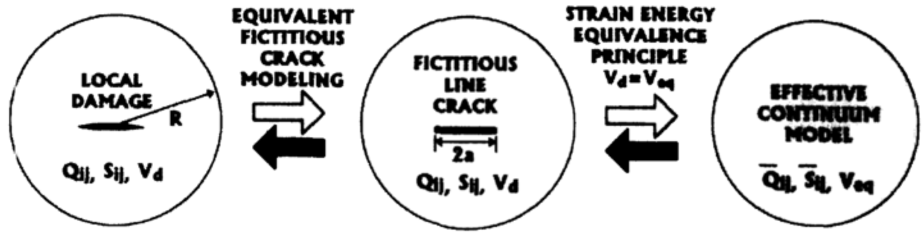
In LDT, as local damage can be characterized by its current geometry, growth direction, and progress of evolution, it was modeled as an equivalent *fictitious* elliptical microcrack. Thus the aspect ratio and orientation of an equivalent elliptical microcrack may approximately represent the current geometry and growth direction of local damage. Since the strain energy for a cracked solid with an elliptical crack is available from fracture mechanics (Sih and Liebowitz, 1967), for both plane problems and three-dimensional problems, the equivalent microcrack was considered as a construct which relates some general damage to effective continuum elastic properties of a damaged isotropic solid on the basis of SEEP. We also benefit from the equivalent microcrack representation, in that the effective characteristics of local damage can be identified from local stresses and strains available in the course of damage analysis. As noted out by Lemaitre(1986), a crack edge may be considered as a local process zone in which damage increases until complete local failure of material occurs. This local approach may be considered as a continuous version of crack propagation. The combination of representing local damage as an effective elliptical microcrack, a consistent damage evolution equation, and the determination of effective continuum elastic properties (into which the local damage is smeared smoothly) by using SEEP may yield a simple, yet powerful new local approach to crack propagation analysis.

As mentioned previously, most current theories of continuum damage mechanics including LDT have been developed for initially *isotropic* solids. However, composite materials are increasingly being used for structural applications where high strength-to-weight and stiffness-to-weight ratios are required. Composite materials in general show anisotropic behavior. Thus, the purpose of this paper is to develop a continuum damage mechanics for anisotropic solids, by extending LDT previously developed for *isotropic* solids.

2. Modeling of Local Damages

In the development of an equivalent continuum model of damaged anisotropic solids, the principles and modeling procedure introduced in LDT will be used in this paper. Figure 1 shows the general features of the continuum damage mechanics for *anisotropic* solids developed in this paper.

Consider a damaged MVC that contains a single microcrack, *i. e.,* local damage. As shown in Fig. 1, a MVC will be modeled as an ECM by determining its effective continuum elastic properties on the basis of SEEP. First assume that damaged MVC and ECM take identical global displacements on their boundaries at the characteristic radius of R (Kassir and Sih, 1967; Lee *et al.,* 1997). This implies that the macro-behavior represented by ECM is the same as that of the damaged solid. Secondly the characteristic size (2a) of local damage is assumed to be relatively small compared to the characteristic size (R) of MVC. This assumption is appropriate in that the effects of neighboring cracks decay rapidly with distance (Rice, 1968) and the complete local failure of solid generally occurs far before the damage variable reaches its maximum value that implies absence of material (Lemaitre, 1985; Krajcinovic, 1989). Hence, local damage within



(a) A local damage within a material volume cell (MVC) (b) Equivalent fictitious line crack modeling of local damage (c) Effective continuum model (ECM) representation based on strain energy equivalence principle (SEEP)

⇨ Damage modeling ⬅ Damage Identification

Fig. 1 General features of the present theory of continuum damage mechanics

each damaged MVC may be approximated as that in an infinite solid. Furthermore, if needed, the stresses on the boundary of the damaged MVC may be replaced by the corresponding strains obtained from the stress-strain relations of the *undamaged* solid. Accordingly the crack energies by Sih and Liebowitz (1967), originally derived in terms of the stresses at infinity, can be used to determine the effective continuum elastic properties of ECM.

As shown in Fig. 1, SEEP may provide the effective continuum elastic properties of ECM by equating the strain energy V_d contained in a damaged MVC to the strain energy V_{eq} in an ECM. That is,

$$V_{eq}(\bar{S};\sigma)=V_d(S,\ D;\ \sigma) \tag{1a}$$

or

$$V_{eq}(\bar{C};\varepsilon)=V_d(C,\ D;\ \varepsilon) \tag{1b}$$

where S and C represent the elastic compliance and stiffness, respectively, for the undamaged state of three-dimensional anisotropic solids, $\bar{S}$ and $\bar{C}$ represent the effective continuum elastic compliance and stiffness for the damaged state. In the preceding equations, D represents the scalar damage variable. Equation (1a) may be used when stresses are specified on the boundaries of the damaged MVC and ECM, while Eq. (1b) is used when the corresponding strains are specified on the same boundaries. The constitutive equations for coupled elasticity and damage may be obtained by simply replacing the undamaged elastic properties by the effective continuum elastic properties for the damaged state.

The strain energy V_d contained in a damaged MVC may be available from fracture mechanics or, alternatively, conventional stress analysis for certain crack problems. Most fracture surfaces can be considered as the continual propagation and coalescence of thin film-like microcracks. Thus it may be pertinent to represent local damage as a line through-crack (simply, line crack), rather than an elliptical through-crack, for plane problems. Thus, in this paper, discussion will be confined to anisotropic plane problems with line cracks. The present study can be readily generalized to other crack problems including three-dimensional problems.

2.1 Effective continuum elastic compliance and scalar damage variable

Consider a two-dimensional anisotropic solid lying in the 1-2 plane. For damaged states, the line crack is assumed to be aligned in the direction "1", as illustrated in Fig. 2. The state of generalized plane stress will be assumed and the stress-strain relation for an undamaged solid may be written in terms of the (undamaged) anisotropic elastic compliance $S_{ij}(i,\ j=1,\ 2,\ 6)$ as

$$\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_6\end{Bmatrix}=\begin{bmatrix}S_{11} & S_{12} & S_{16}\\ S_{12} & S_{22} & S_{26}\\ S_{16} & S_{26} & S_{66}\end{bmatrix}\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \sigma_6\end{Bmatrix} \text{ or } \{\varepsilon\}=[S]\{\sigma\} \tag{2}$$

while, for the ECM of damaged solid, it is written in terms of the damaged (or effective continuum) elastic compliance $\bar{S}_{ij}(i,\ j=1,\ 2,\ 6)$ as

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} \text{ or } \{\varepsilon\} = [\bar{S}]\{\sigma\} \quad (3)$$

The effective continuum elastic compliance $\bar{S}_{ij}$ is to be determined, by using SEEP, in terms of the undamaged elastic compliance S_{ij} and a single scalar damage variable D that is defined in the following. In Eqs. (2) and (3), the contracted notations for stresses and stains have been used for plane problems, *i. e.*, $\sigma_1 = \sigma_{11}$, $\sigma_{2}, = \sigma_{22}$, $\sigma_6 = \tau_{12}$, $\varepsilon_1 = \varepsilon_{11}$, $\varepsilon_2 = \varepsilon_{22}$, and $\varepsilon_6 = \gamma_{12}$.

The components of elastic compliance can be represented in terms of engineering constants (*i. e.*, Young's moduli, shear moduli, and Poisson's ratios) (Lekhnitskii, 1963). They are, for anisotropic solids:

$$S_{11} = \frac{1}{E_1}, \quad S_{22} = \frac{1}{E_2}, \quad S_{66} = \frac{1}{G_{12}},$$
$$S_{12} = -\frac{\nu_{12}}{E_1}, \quad S_{16} = -\frac{\nu_{16}}{G_{12}}, \quad S_{26} = -\frac{\nu_{26}}{G_{12}} \quad (4a)$$

for orthotropic solids:

$$S_{11} = \frac{1}{E_1}, \quad S_{22} = \frac{1}{E_2}, \quad S_{66} = \frac{1}{G_{12}}, \quad S_{12} = -\frac{\nu_{12}}{E_1},$$
$$S_{16} = 0, \quad S_{26} = 0 \quad (4b)$$

and, for isotropic solids:

$$S_{11} = S_{22} = \frac{1}{E}, \quad S_{66} = \frac{1}{G} = \frac{2(1+\nu)}{E},$$
$$S_{12} = -\frac{\nu}{E}, \quad S_{16} = 0, \quad S_{26} = 0 \quad (4c)$$

Consider an *undamaged* MVC of anisotropic solid, which is subjected to boundary stresses σ at the characteristic radius of R. The strain energy contained in the undamaged MVC may be written in the form (Sih and Liebowitz, 1967)

$$V_o = \frac{1}{2}\pi R^2 \{\sigma\}^T [S] \{\sigma\} \quad (5)$$

Similarly, the strain energy stored in an ECM of *damaged* anisotropic solid subjected to the same boundary stresses may be written in terms of the effective continuum elastic compliance $\bar{S}_{ij}$ as

$$V_{eq} = \frac{1}{2}\pi R^2 \{\sigma\}^T [\bar{S}] \{\sigma\} \quad (6)$$

A damaged solid may not store as much strain energy under a given deformation as an undamaged solid because of the degradation of elastic moduli. The change of strain energy storage capacity of a two-dimensional anisotropic solid due to the presence of a line crack is required to derive the effective continuum elastic compliance $\bar{S}_{ij}$ on the basis of SEEP. The strain energy released in forming a crack is often called the crack energy. The crack energy for generating a line crack of length 2a in an infinite two-dimensional anisotropic solid was derived by Sih and Liebowitz (1967), and it can be expressed as

$$V_c = \frac{1}{2}\pi a^2 \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & S_{22}\delta_{22} & 0 \\ 0 & 0 & S_{11}\delta_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} \quad (7)$$

with

$$\delta_{22} = (\alpha_1^2 + \beta_1^2)\beta_2 + (\alpha_2^2 + \beta_2^2)\beta_1, \quad \delta_{66} = \beta_1 + \beta_2 \quad (8)$$

where α_i and $\beta_i (i = 1, 2)$ are determined from the four roots $r_{1,2} = \alpha_1 \pm i\beta_1$ and $r_{3,4} = \alpha_2 \pm i\beta_2$ of an algebraic equation given as

$$S_{11}r^4 - 2S_{16}r^3 + (3S_{12} + S_{66})r^2 - 2S_{26}r + S_{22} = 0 \quad (9)$$

For orthotropic crack systems in which the line crack is aligned with one plane of material symmetries of orthotropic solids, two values δ_{22} and δ_{66} in Eq. (7) or Eq. (8) can be expressed in terms of engineering constants as follows:

$$\delta_{22} = \sqrt{2}\left[\left(\frac{E_2}{E_1}\right)^{1/2} + \frac{E_2}{2G_{12}} - \nu_{12}\frac{E_2}{E_1}\right]^{\frac{1}{2}} \quad (10)$$
$$\delta_{66} = \sqrt{2}\left[\left(\frac{E_1}{E_2}\right)^{1/2} + \frac{E_1}{2G_{12}} - \nu_{12}\right]^{\frac{1}{2}}$$

The two values δ_{22} and δ_{66} can now be further simplified for the case of isotropic solids as

$$\delta_{22} = \delta_{66} = 2 \quad (11)$$

The strain energy V_d that can be stored in a damaged MVC may be given in general form as

$V_d = V_o + V_c$ (when boundary traction is specified) (12a)

or

$V_d = V_o - V_c$ (when boundary displacement is specified) (12b)

depending on whether the traction or displacement is specified on the boundary of a damaged MVC, because the crack energies by specifying tractions and displacements are numerically equal but differ in sign (Kassir and Sih, 1967). This is in agreement with a general result established by Spencer (1965) using Betti's reciprocal theorem.

Using the energies given by Eqs. (5) and (7), the strain energy V_d that can be stored in a damaged MVC is obtained, from (12a), as follows:

$$V_d=\frac{1}{2}\pi R^2\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \sigma_6\end{Bmatrix}^T\begin{bmatrix}S_{11} & S_{12} & S_{16}\\ S_{12} & S_{22}(1+\delta_{22}D) & S_{26}\\ S_{16} & S_{26} & S_{66}(1+S)\,\delta_{66}D/S_{66}\end{bmatrix}\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \sigma_6\end{Bmatrix} \tag{13}$$

$$=\frac{1}{2}\pi R^2\{\sigma\}^T[S_{ij}(1+\bar{e}_{ij}D)]\{\sigma\}$$

$(i,\ j=1,\ 2,\ 6;\text{no sum})$

where $\bar{e}_{ij}$ are defined as

$$\bar{e}_{22}=\delta_{22},\ \bar{e}_{66}=\frac{S_{11}}{S_{66}}\delta_{66},\ \text{and other } e_{ij}=0 \tag{14}$$

The parameters $\bar{e}_{ij}$ generally depend on the elastic compliance of the undamaged solid as well as on the crack orientation, since they are calculated by assuming that the line crack is aligned with the direction "1", one of local coordinates. In Eq. (13), D is a scalar variable defined as

$$D=\left(\frac{a}{R}\right)^2 \tag{15}$$

In this paper, the scalar variable D will be considered as a new definition of a damage variable. This new scalar damage variable may be interpreted as the ratio of the effective damaged area (πa^2) to the total area of the characteristic region considered (πR^2), which differs somewhat from the classical damage variable D used in typical theories of continuum damage mechanics (*e. g.*, Lemaitre, 1992).

On the basis of SEEP, represented by Eq. (1a), we equate Eq. (6) to Eq. (13) in order to derive the effective continuum elastic compliance $\bar{S}_{ij}$ in a simple form as

$$\bar{S}_{ij}=S_{ij}(1+\bar{e}_{ij}D)\ (i,\ j=1,\ 2,\ 6;\text{no sum}) \tag{16}$$

Equation (16) implies that initially anisotropic solids show anisotropic behavior after they are damaged, while initially orthotropic solids (in which one plane of material symmetries is aligned with the line crack) and isotropic solids show orthotropic behaviors after damage. The effective engineering constants of ECM are derived from Eq. (16) and listed in Table 1 for initially anisotropic, orthotropic, and isotropic solids.

Table 1 Effective continuum engineering constants for generalized plane stress problems, depending on the type of material symmetry.

Effective Effective engineering constraints	Type of initially undamaged solids		
	Anisotropic	Orthotropic	Isotropic
$\bar{E}_1$	E_1	E_1	E
$\bar{E}_2$	$\frac{E_2}{1+\bar{e}_{22}D}$	$\frac{E_2}{1+\bar{e}_{22}D}$	$\frac{E}{1+2D}$
$\bar{G}_{12}$	$\frac{G_{12}}{1+\bar{e}_{66}D}$	$\frac{G_{12}}{1+\bar{e}_{66}D}$	$\frac{G}{1+D/(1+\nu)}$
$\bar{\nu}_{12}$	ν_{12}	ν_{12}	ν
$\bar{\nu}_{16}$	$\frac{\nu_{16}}{1+\bar{e}_{66}D}$	-	-
$\bar{\nu}_{26}$	$\frac{\nu_{26}}{1+\bar{e}_{66}D}$	-	-

2.2 Effective continuum elastic stiffness

The strain energies contained in an *undamaged* MVC of anisotropic solids and an ECM of damaged MVC, both of which are subject to the same boundary strains ϵ at the characteristic radius of R, may be written in terms of the reduced stiffness $Q_{ij}(i,\ j=1,\ 2,\ 6)$ and the effective continuum elastic stiffness $\bar{Q}_{ij}$, respectively, as

$$V_o=\frac{1}{2}\pi R^2\{\varepsilon\}^T[Q]\{\varepsilon\} \tag{17}$$

$$V_{eq}=\frac{1}{2}\pi R^2\{\varepsilon\}^T[\bar{Q}]\{\varepsilon\} \tag{18}$$

where the reduced stiffness Q_{ij} are obtained from the stiffness C_{ij} for three-dimensional solids as follows (Tsai, 1988):

$$Q_{ij}=C_{ij}-\frac{C_{i3}C_{j3}}{C_{33}}(i,\ j=1,\ 2,\ 6) \tag{19}$$

When strains are specified on the boundary of a damaged MVC, the strain energy V_d that can be stored in the damaged MVC may be obtained from Eq. (12b) by using the same crack energy V_c given by Eq. (7). In this case, the stresses appearing in Eq. (7) should be replaced by the equivalent strains calculated from the stress -strain relations for undamaged anisotropic solids. This may result in

$$V_d = \frac{1}{2}\pi R^2 \{\varepsilon\}^T [Q_{ij}(1 - e_{ij}D)] \{\varepsilon\}$$

$$(i,\ j = 1,\ 2,\ 6;\ \text{no sum}) \tag{20}$$

By equating Eq. (18) to Eq. (20) on the basis of SEEP, the effective continuum elastic stiffness $\bar{Q}_{ij}$ are obtained as

$$\bar{Q}_{ij} = Q_{ij}(1 - e_{ij}D)\ (i,\ j = 1,\ 2,\ 6;\ \text{no sum}) \tag{21}$$

where the parameters e_{ij} are listed in Table 2. It is proved from Table 2 that the parameters e_{ij} for initially isotropic solids are identical to that given in Lee's damage theory for isotropic solids (Lee et *al.*, 1997).

Table 2 Values of parameters e_{ij} depending on the type of material symmetry.

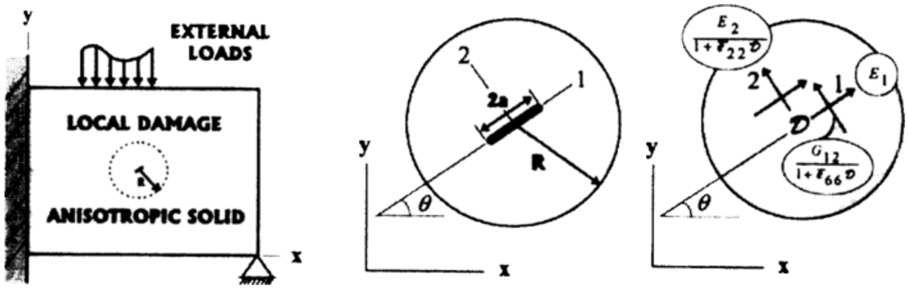
Parameters (e_{ij})	Type of initially undamaged solids		
	Anisotropic	Orthotropic	Isotropic
e_{11}	$\frac{Q_{12}^2}{Q_{11}} S_{22} \bar{e}_{22} + \frac{Q_{16}^2}{Q_{12}} S_{66} \bar{e}_{66}$	$\left(\frac{\nu_{12}^2}{1 - \nu_{12}\nu_{21}}\right) \frac{E_2}{E_1} \bar{e}_{22}$	$\frac{2\nu^2}{1 - \nu^2}$
e_{22}	$Q_{22} S_{22} \bar{e}_{22} + \frac{Q_{26}^2}{Q_{22}} S_{66} \bar{e}_{66}$	$\left(\frac{1}{1 - \nu_{12}\nu_{21}}\right) \bar{e}_{22}$	$\frac{2}{1 - \nu^2}$
e_{66}	$\frac{Q_{26}^2}{Q_{66}} S_{22} \bar{e}_{22} + Q_{66} S_{66} \bar{e}_{66}$	$\bar{e}_{66}$	$\frac{1}{1 + \nu}$
e_{12}	$Q_{22} S_{22} \bar{e}_{22} + \frac{Q_{16} Q_{26}}{Q_{12}} S_{66} \bar{e}_{66}$	$\left(\frac{1}{1 - \nu_{12}\nu_{21}}\right) \bar{e}_{22}$	$\frac{2}{1 - \nu^2}$
e_{16}	$\frac{Q_{12} Q_{26}}{Q_{16}} S_{22} \bar{e}_{22} + Q_{66} S_{66} \bar{e}_{66}$	-	-
e_{26}	$Q_{22} S_{22} \bar{e}_{22} + Q_{66} S_{66} \bar{e}_{66}$	-	-

3. Identification of Local Damages

In the preceding section, the elastic behavior of a damaged solid was represented by the effective continuum elastic compliance $\bar{S}_{ij}$ or stiffness $\bar{Q}_{ij}$ by modeling local damage as an equivalent fictitious line crack. In effect, a solid with embedded damage (equivalently, line cracks) is smeared smoothly into an equivalent continuum with elastic compliance $\bar{S}_{ij}$ or stiffness $\bar{Q}_{ij}$. By replacing all damaged local zones by their equivalent *anisotropic* continua, conventional methods may be employed for further stress and damage analyses. This continuum approach seems useful only when the current state of local damage (*i. e.*, damage size and damage growth direction) is known, as that information is required to calculate the effective continuum elastic properties. Unfortunately, in practice, it is perhaps impossible to identify the damage state in detail because the embedded local damage is inaccessible. Hence, it may require an innovative method by which the local damage state can be identified.

Since local damage is modeled as an equivalent line crack, the identification of local damage is equivalent to identifying the current size and growth direction of a line crack. As the size of current local damage is measured by the damage variable D, which is determined by the damage evolution equation (see Eq. (25)), the present discussion will be confined to the identification of the current crack orientation (θ) with respect to the global coordinates ($x - y$), as shown in Fig. 2. Once the current crack orientation is identified, the corresponding damaged zone can be converted to an effective continuum by using Eq. (16) or (21). Since the effective continuum elastic properties are obtained from Eq. (16) or (21) with respect to the crack (local) coordinates (1-2), they should be transformed back to the global coordinates and used in the next step of an incremental calculation process.

As can be seen from Table 1 and also from Fig. 2(c), the effective Young's modulus $\bar{E}_1$ in the "1" direction of crack coordinates (1-2) does not change due to the presence of the line crack, but



(a) Current state of a local damage within a structure (b) Equivalent line crack representation (c) Effective engineering constants for effective continuum model

Fig. 2 Identification of local damage

the effective Young's modulus $\bar{E}_2$ in the "2" direction and the effective shear modulus $\bar{G}_{12}$ are reduced in magnitude by factors of $(1+\bar{e}_{ii}D)$ $(i=2, 6)$. As stress analysis is typically conducted in the course of an incremental damage analysis, it may be desirable to use the current values of stresses and strains at a damaged local point to determine the crack orientation. Though there can be some alternate approaches to determine the crack orientation, an iterative approximate approach will be introduced herein.

To determine the current crack orientation θ, assume that D (current damage variable) as well as σ_I and ε_J (current stresses and strains with respect to the global coordinates) are known at a local damage. After estimating the crack orientation with respect to the global coordinates, say θ_0, the coordinate transformation is used to compute the stresses and strains with respect to the estimated crack coordinates, *i. e.*, σ_i and ε_j. The effective elastic moduli for the damaged state are then estimated from σ_i and ε_j as follows:

$$E_1^*=\frac{\sigma_1}{\varepsilon_1},\quad E_2^*=\frac{\sigma_2}{\varepsilon_2},\quad G_{12}^*=\frac{\sigma_6}{\varepsilon_6} \tag{22}$$

For undamaged state, the compliance S_{IJ} with respect to the global coordinates is known in advance. Thus the compliance S_{ij} with respect to the estimated crack coordinates can be readily computed from S_{IJ} by a coordinates transformation. Using the compliance S_{ij}, the parameters $\bar{e}_{22}$ and $\bar{e}_{66}$, and the elastic moduli, E_1, E_2, and G_{12} for undamaged state can be computed. If the estimated crack orientation θ_0 is correct, then the following three relations from Table 1 may be satisfied:

$$\frac{E_1^*}{E_1}=1,\quad \frac{E_2^*}{E_2}=\frac{1}{1+\bar{e}_{22}D}<1,\quad \frac{G_{12}^*}{G_{12}}=\frac{1}{1+\bar{e}_{66}D}<1 \tag{23}$$

However, the estimated crack orientation θ_0 may not exactly satisfy all three relations in Eq. (23) at once. Hence, the best estimate of θ may be found from an error minimizing approach as:

$$\text{Minimize } f(\theta)=\left(\frac{E_1^*}{E_1}-1\right)^2+\left(\frac{E_2^*}{E_2}-\frac{1}{1+\bar{e}_{22}D}\right)^2+\left(\frac{G_{12}^*}{G_{12}}-\frac{1}{1+\bar{e}_{66}D}\right)^2 \tag{24}$$

4. An Iterative Approach for Damage Analysis

The scalar damage variable D defined by Eq. (15) is identical to that defined in LDT. Thus, the damage evolution equation in LDT can be used in damage analysis (Lee *et al.*, 1997):

$$\dot{D}=\eta D^{(0.5+0.25N)}\bar{\sigma}^N H(\sigma_{eq}-\sigma_{TH}) \tag{25}$$

where η is a material constant, N is a parameter defined in Paris's crack growth law (1963), σ_{eq} is the von Mises equivalent stress, σ_{TH} is a threshold stress above which damage will grow, and H is the Heaviside step function. In Eq. (25), $\bar{\sigma}$ represents the so-called damage equivalence stress defined by Lemaitre (1992).

In the previous sections, current values of the damage variable, stresses, and strains at local damage are assumed available in order to identify the current orientation of a fictitious line crack. As the current stresses and strains should be consistent with the current effective continuum elastic properties, an iterative numerical approach is considered herein to enforce this consistency condition at each end of time increment during a damage analysis. It may be summarized as follows:

(1) Use the data of the *previous* damage state (i. e., effective continuum elastic properties and damage variable at the previous time step) to predict current (local) stresses and strains with respect to global coordinates by using a conventional stress analysis method .

(2) Use the damage identification procedure of Section 3 to predict the current crack orientation θ with respect to global coordinates.

(3) In parallel with step 2, the damage evolution equation is solved to predict the current damage variable by using the stresses and strains predicted in step 1 and previous damage variable.

(4) Predict current effective continuum elastic properties, from Eq. (16) or (21), with respect to the crack coordinates predicted in step 2. Use the coordinate transformation to calculate current effective continuum elastic properties with respect to global coordinates.

(5) Replace the *previous* damage state with the *predicted* current damage state, and then repeat the above procedures to obtain improved predictions of the current damage state.

(6) After sufficient convergence of predicted current damage state, go back to the step 1 with a time increment for next iteration.

Though a specific iterative numerical approach is introduced herein, there should be undoubtedly other alternate numerical approaches to predict the current damage state in consistent ways.

5. Conclusions

As anisotropic materials such as composite materials are increasingly being used for structural applications, a theory of continuum damage mechanics for *anisotropic* solids is introduced by using the crack energy from fracture mechanics. The material behavior of damaged solids is modeled by the effective continuum elastic properties, into which local damage is smoothly smeared, by using two major concepts: the strain energy equivalence principle and the equivalent line crack representation of a local damage. We benefit from these concepts, in that current local damage state can be identified from the local stresses and strains available in the course of damage analysis. Thus, combining the damage modeling process with the damage identification process may provide a simple, unified, and yet powerful continuum approach to predict failures of structural and mechanical components.

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